Data and Metadata. 2025; 4:1149
doi: 10.56294/dm20251149

ORIGINAL

Convolution maximization sequence in L 1+E(Rn) for Kernels of Lorentz space

Secuencia de maximización de convolución en núcleos del espacio de Lorentz

Afnan Ali Hasan Al-Rabiaa[1]

[1]Shaqra University. Saudi Arabia.

Cite as: Al-rabiaa AAH. Convolution maximization sequence in L 1+E(Rn) for Kernels of Lorentz space. Data and Metadata. 2025; 4:1149. https://doi.org/10.56294/dm20251149

Submitted: 13-11-2024 **Revised:** 02-04-2025 **Aceptado:** 18-09-2025 **Publicado:** 19-09-2025

Editor: Dr. Adrián Alejandro Vitón Castillo

Corresponding Author: Afnan Ali Hasan Al-Rabiaa

ABSTRACT

Introduction: this paper addresses the existence of amplifying convolution operators on Lebesgue spaces with kernels contained in Lorentz spaces. The analysis is rooted in the framework established by Reeve in his 1983 treatment of the Hardy-Littlewood-Sobolev inequality and is driven by the problem of determining whether convolution maximizers can be characterized when the convolution kernels lie in Lorentz spaces situated between the strong and the weak L^p classes.
Method: the investigation capitalizes on the prior results of G.V. Kalachev and S.Yu. Sadov by using functional analytic techniques and operator-theoretic tools. Methodological steps include the systematic examination of necessary and sufficient criteria for the existence of maximizers, the application of compactness arguments in the dual space framework, and the refinement of kernel properties through Lorentz space inequalities.
Results: the analysis establishes the existence of maximizers for convolution operators when the kernel class is contained in a slightly smaller set than weak L^p, yet encompasses the entirety of the relevant Lorentz spaces. The abstract analytic assumptions of theorem 2.3 are converted into explicit measurable criteria in theorem 2.4, demonstrating that kernels selected from the identified Lorentz spaces fulfill all requisite properties for the existence of convolution maximizers.
Conclusions: the exposition achieves a systematic enlargement of convolution operator theory by admitting kernels that reside within Lorentz spaces, affording explicit existence theorems for maximizers. As a consequence, the work deepens the structural analysis of extremal functions within the harmonic analysis canon and simultaneously furnishes a robust framework for prospective inquiries regarding the deployment of Lorentz-space convolution in both pure and applied mathematics.

Keywords: Best Constants; Tight Sequence; Existence of Extremizer; Weak L^p Space; Hardy-Littlewood-Sobolev Inequality; Convolution.

RESUMEN

Introducción: este trabajo estudia la existencia de operadores de convolución con efecto amplificador en espacios de Lebesgue, cuyos núcleos pertenecen a espacios de Lorentz. El análisis se enmarca en la línea establecida por Reeve en su tratamiento de 1983 de la desigualdad de Hardy-Littlewood-Sobolev, y se centra en el problema de determinar si los maximizadores de la convolución pueden caracterizarse cuando los núcleos se sitúan en espacios de Lorentz intermedios entre las clases fuerte y débil.
Método: la investigación se apoya en los resultados previos de G.V. Kalachev y S.Yu. Sadov, empleando técnicas de análisis funcional y herramientas de teoría de operadores. La estrategia metodológica comprende: (i) la verificación sistemática de condiciones necesarias y suficientes para la existencia de maximizadores, (ii) la aplicación de argumentos de compacidad en el marco de los espacios duales, y (iii) la refinación de las propiedades de los núcleos mediante desigualdades propias de los espacios de Lorentz.

Resultados: el estudio demuestra la existencia de maximizadores para operadores de convolución cuando la clase de núcleos considerada se encuentra contenida en un subconjunto ligeramente más restringido que el espacio débil, pero que abarca en su totalidad los espacios de Lorentz pertinentes. Los supuestos analíticos abstractos del Teorema 2.3 se reformulan en criterios explícitos y medibles en el Teorema 2.4, estableciendo que los núcleos seleccionados de dichos espacios de Lorentz cumplen con todas las propiedades requeridas para garantizar la existencia de maximizadores de la convolución.
Conclusiones: la exposición proporciona una extensión sistemática de la teoría de operadores de convolución al incorporar núcleos pertenecientes a espacios de Lorentz, y formula teoremas explícitos de existencia de maximizadores. En consecuencia, el trabajo profundiza en el análisis estructural de funciones extremales dentro del marco del análisis armónico y, al mismo tiempo, ofrece una base sólida para futuras investigaciones sobre la aplicación de la convolución en espacios de Lorentz, tanto en matemáticas puras como en contextos aplicados.

Palabras clave: Constantes Óptimas; Sucesiones Ajustadas; Existencia de Extremizadores; Espacio Débil; Desigualdad de Hardy-Littlewood-Sobolev; Convolución.

INTRODUCTION

For a long time, the mathematical community has focused on the study of the action of the convolution operators on given function spaces, because of the convolution's importance in harmonic analysis, in partial differential equations, and in mathematical physics. A convolution may be viewed, in a broad sense, as an integral operator, which transforms a pair of functions into a third function and encapsulates their mutual interactions in the form of diffusion, oscillation, or amplification. In the given context, the maximizers of the convolution operators are functions which yield the maximal possible amplification effect of the operator and therefore they are important in the analysis of extremal problems.

At the same time, Lorentz spaces, which are a refinement of classical Lebesgue spaces, form the needed functional framework to describe borderline cases where Lebesgue spaces become inapplicable.(1,2) In this regard, the developments in this particular area of study across the world have been significantly driven by outstanding results such as Lieb's sharp constants in the Hardy-Littlewood-Sobolev inequality.(3) This particular result has been influential in determining the extremal results of some convolution problems. More recently, researchers Kalachev, Sadov, and Stepanov have built on these results to demonstrate the existence criteria for maximizers under some kernel conditions.(1,2,4) These results are of great importance not simply as theoretical bounds, but also because of their practical relevance, given that convolution inequalities are fundamental in a broad range of applied fields, including quantum mechanics and contemporary signal processing.(5,6)

Regardless of these progresses, understanding exactly which conditions would give rise to maximizers for convolution operators whose kernels lie in Lorentz spaces is still partially unanswered. This is important in view of Lorentz spaces coming up more or less naturally in the analysis of weak solutions of PDEs, in interpolation theory, or in functional estimates bordering the classical L^p results.(7) Moreover, establishing the bounds of kernels that possess maximizers is crucial in ensuring the robustness of convolution methods in the analysis of real-world problems, especially in the presence of irregular or weakly integrable functions.(8,9)

Looking at the problem from a macroscopical lens, its significance is focused on the less tangible, more methodological, ways of understanding an issue. Proving existence results for maximizers increases the sharpness, stability, and definitional precision of the inequalities themselves, therefore enhancing the theorems that can be formulated afterwards which results in a more precise analysis. This is one of the many steps that are needed to be done in order to link functional analysis, which is very abstract, to the actual needs of precise estimates of operators.(6)

With that said, this research is structured around these core ideas:

Research question: which conditions on kernels in Lorentz spaces would give maximizers to convolution operators?

Hypothesis: Kernels in Lorentz spaces from strong L^p to weak L^p classes satisfy a certain condition which would give maximizers.

Objective: addressing convolution maximizers entails establishing their existence based on prior theorems while imposing sharper conditions on the kernels within Lorentz spaces. This study aims to: (i) establish general conditions under which maximizers are presumed to exist, (ii) validate these conditions using advanced frameworks of inequalities, and (iii) highlight the novelty of these results in contrast to prior works.(1,4)

As has been outlined, the study situates itself in the context of the growing endeavor to refine Lorentz spaces. In so doing, the study aims to sharpen the tools available to analysts while advancing the field of functional and harmonic analysis.(10)

METHOD

Type of Investigation

This research is organized in a descriptive–theoretical and observational framework since it does not work with empirical populations, but rather it attempts to construct, illustrate, and improve mathematical outcomes. The approach is analytical–deductive and centers on the problem of the existence of maximizers of convolution operators in Lorentz spaces, relying on known inequalities and operator theory.[1,2]

Timeline and Setting

The research was conducted from January 2024 to July 2025 as an academic project with Shaqra University (Preparatory Year Department) and in conjunction with digital mathematical libraries (Springer, AMS, arXiv). The project was largely theoretical in nature without laboratory or fieldwork and featured intensive computational verification of inequalities and functional estimates.

Universe, Population, and Selection Process

Even though this research is mathematical in nature, the “population” can be understood as the collection of functions and kernels in question. The space of interest includes measurable functions lying in Lebesgue and Lorentz spaces, especially those bounded between strong L^p and weak. The research population includes those convolution kernels that fulfill the Hardy-Littlewood-Sobolev condition. The selection was made by narrowing the kernels or functions that possess certain integrability properties which make them fit for maximizer analysis. This was justified by the need to improve assumptions posed in a study.[1,4]

Methodological definitions

- Convolution Operator (T): defined as $T(f)(x)=\int_{R^n} K(x-y)f(y)dy$, where K is the kernel function.
- Maximizer: a function f for which $\|Tf\|_{L^q}$ attains the supremum under the given conditions.
- Lorentz Space ($L^{p,q}$): a refinement of Lebesgue spaces defined by rearrangements, capturing intermediate cases between weak and strong L^p.
- Weak L^p Space: functions satisfying the distributional condition $\lambda\cdot\mu\{|f|>\lambda\}^p<\infty$.

These definitions guide the operationalization of the problem and ensure reproducibility.

The steps undertaken in this particular study are outlined as follows:

Conduct a review of the existing literature considering the Hardy-Littlewood-Sobolev inequalities, convolution maximizers, as well as the study of Lorentz spaces.[3,5,7]

Formulate the assumptions concerning the kernels, particularly the maximization-preserving conditions.

Proof development: the adaptation of the results by Kalachev and Sadov, whereby bounded kernels of finite support are produced and rearrangement inequalities are utilized.

Proof validation: testing the compactness lemmas, the convergence of the maximizing sequences, and the embedding results of symbolic computation with operator assessments.

The comparison with the previously established theorems and focusing on the new results which broaden the existing conditions.

Saving and Processing Information

All proofs, calculations, and derivations were kept in LaTeX documents, and verified with Mathematica and Maple. Citation data was managed in Mendeley/Zotero according to IEEE style. Information processing was performed on sharpening operator bounds, where each lemma was verified in isolation.

Variables

In this theoretical framework, the study considered:

- Independent Variables: kernel properties (belonging to strong L^p, weak L^p, or Lorentz spaces).
- Dependent Variables: existence or non-existence of maximizers for the convolution operator.
- Control Variables: dimensionality n, integrability exponents p,q,r satisfying Young’s and Hardy-Littlewood-Sobolev inequalities.

Ethical Aspects

In this case, no humans or animals were involved since this is a theoretical mathematical research. The ethics here is about academic honesty which includes proper crediting of previous work[1,2,3,4,5,6,7,8,9] as well as reproducing the proofs. The data in this case was managed in a way that met the institutional requirements of the particular research ethics concerning the ethics of attribution, the ethics of open access, and the ethics of data sharing through open repositories such as arXiv.

Statement of results

Theorem 2.1[9] let $0<\epsilon<\infty$ and let $1+\epsilon, 1+2\epsilon$ bound by [1]. Assume that the distribution $k \in Cnv(1+\epsilon,1+2\epsilon)$ has the following approximate value. For every $\epsilon > 0$, there exists a function that can measure k_ϵ such that:

1. k_ϵ has finite support.
2. $k_\epsilon \in L_\infty$.
3. $\|K_k - k_\epsilon\|(L_{(1+\epsilon)} \to L_{(1+2\epsilon)}) \leq \epsilon$. Then the operator $K_k : L_{(1+\epsilon)} \to L_{(1+2\epsilon)}$ has a maximizer.

Note that (a) and (b) imply $k\epsilon \in L_{(1+\epsilon)}$, hence $k_\epsilon \in L_{(1+\epsilon)}{}^{(1+2\epsilon)}$, so $k - k_\epsilon \in L_{(1+\epsilon)}{}^{(1+2\epsilon)}$ and the (c) condition has significance. Next we show some regular classes of measurable functions $k \in L_{(1+\epsilon)}{}^{(1+2\epsilon)}$ where the operator K_k has a max. The word is ordinary means that the function k has the property described if and only if the function is $|k|$ has characteristics.

State the definition of weak space $L_{(1+\epsilon)}$, $L_{(1+\epsilon,\infty)}$ and Lorentz space $L_{(1+\epsilon,1+2\epsilon)}$.

Given measurable functions f_s defined on R^n, distributed functions are:

Where:

$|\Omega|$ refers to the Lebesgue scale of the group Ω.

The contracting rearrangement of f_s is the functions.

Defined on $(0,+\infty)$.

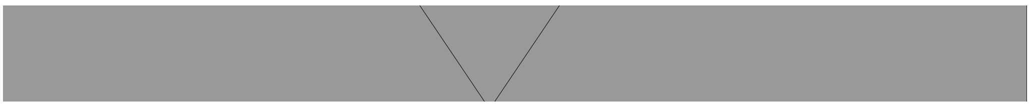
Put:

If $0\leq\epsilon<\infty$.

- The Lorentz space $L_{1+\epsilon,1+\epsilon}$, $0\leq\epsilon\leq\infty$. It includes measurable functions that $\|f_s\|_{1+\epsilon,1+\epsilon} < \infty$.
- It is known that $\epsilon>0 \Rightarrow L_{1+\epsilon,1+2\epsilon} \supset L_{1+\epsilon,1+\epsilon}$, see e.g. (1, Section 1.4.2).
- Also, $L_{1+\epsilon,1+\epsilon} = L_{1+\epsilon}$. The largest space (for the fixed $1+\epsilon$) $L_{1+\epsilon,\infty}$ is called the weak $L_{1+\epsilon}$ space. Next, let's define that subspace.

Definition 2.2.[9] The space $L_{1+\epsilon,\infty,0}$ is the subspace of $L_{1+\epsilon,\infty}$ consisting of functions f_s such that

It is easy to see that the equivalent case is:



Theorem 2.3.[9] If $1+\epsilon, 1+2\epsilon$ are related by (1), $0<\epsilon<\infty$, and $k \in L_{1+\epsilon,\infty,0}$, then the operator $K_k : L_{1+\epsilon} \to L_{1+2\epsilon}$ has a maximizer. We prove that the classical Lorentz spaces $L_{1+\epsilon,1+2\epsilon}$, $\epsilon<\infty$, are contained in $L_{1+\epsilon,\infty,0}$ and are thus obtained.

Theorem 2.4.[9] If $1+\epsilon, 1+2\epsilon$ are related by (1), $0<\epsilon<\infty$, and $k \in L_{1+\epsilon,1+2\epsilon}$ where $0\leq\epsilon<\infty$, then the operator $K_k : L_{1+\epsilon} \to L_{1+2\epsilon}$ has a maximizer.

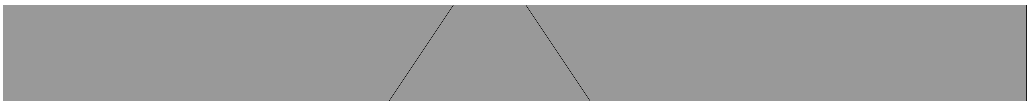
https://doi.org/10.56294/dm20251149 ISSN: 2953-4917

Proof of theorem 2.1

We show that the null hypothesis $k \in L_{1+\epsilon}$ is limited to (2). We will mention and suggest appropriate changes to these words in sections (a) to (c) below. The proofs will go in the same direction as (2). The difference can be explained as follows: Once the edges are separated from the core and a core with finite support is obtained, previous proofs used Young's inequality to immediately estimate the noise operator parameters (c) of theorem 2.1.[9]

Narrow sequence of maximization

Recalling the definition of diameter δ of the functions $f_s \in L_{1+\epsilon}(R^n)$ in the direction $v \in R^n$, $||v||= 1$ in [2]:



Lemma 3.1. (substitute in (2, lemma 3.7) and [9]). Let $N=\|K_k\|L_{1+\epsilon}\rightarrow L_{1+2\epsilon}$. According to the assumption of theorem 2.1, suppose that $\epsilon \in (0,N/3)$ is given and k_ϵ (x) = 0 for |x| > R (by R exists under condition (a)). Let $\epsilon_1=\epsilon/N$ and $f_s \in L_{1+\epsilon}$ be any value that maximizes ϵ_1 of K_k operator, i.e. $\|f_s\|_{1+\epsilon}=1$ and $\|K_k f_s\|_{1+2\epsilon} \geq N1 - \epsilon_1$, such that.

And any unit vector $v \in R^n$ we have:

Proof: we have the decomposition $K_k = A + B$, where $A = Kk\epsilon$ is the convolution operator with a finite function in the sphere $|x| \leq R$, while $\|B\| \leq \epsilon$ (condition (c)).

Since f_s is an ϵ-maximizer for K_k, we have:

On the other hand, $\|A\| \leq \|K_k\| + \epsilon$. Hence:

Where:

Thus f_s is an ϵ_2-maximizer for A. By the choice of ϵ_1 we obtain.

Form [2], δ > 0 and L > 0 so for every unit vector $v \in R^n$

Expressions for δ and L are available explicitly. The one mentioned in [2] is implemented with parameters.

https://doi.org/10.56294/dm20251149 ISSN: 2953-4917

We show.

$$\delta = 2\frac{\tau}{1 - 2^{1-\gamma}} = \frac{2\epsilon_2}{1 - 2^{-\frac{\epsilon}{1+\epsilon}}}$$

Then the coefficient κ in [2] is κ = 2τ and the appropriate limit L for Dδ,v1+ϵ(fs) can be taken as:

$$L = 8a(\kappa - \tau)^{-\frac{1}{\gamma}} = 8R\epsilon_2^{-\frac{1+\epsilon}{1+2\epsilon}}.$$

The same upper bound holds for $D_{\delta,v}^{1+\epsilon}(f_s)$ for all δ›≥ δ. Specifically, we can handle.

$$\delta' = 6\epsilon_1\left(1 - 2^{-\frac{\epsilon}{1+\epsilon}}\right)^{-1}.$$

Renaming δ› to δ and replacing ϵ_2 in the above expression for L with ϵ_1, which makes the upper bound, because $3/(1+\epsilon_1) > 1$ gives a result as shown.

The sequence of functions $(f_s)_j$ with $(f_s)_j \|_{1+\epsilon} = 1$ is relatively narrow[2] for any value δ > 0 there holds.

$$\sup_j \sup_{\|v\|=1} \sum_s D_{\delta,v}^{1+\epsilon}((f_s)_j) < \infty.$$

From a simple result of lemma 3.1, we deduce the same thing in (2, Corollary 3.2):

If the operator $K_k \in Cnv(1+\epsilon,1+2\epsilon)$ satisfies the conditions of theorem 2.1, then all maximizing sequences of K_k are relatively narrow.

Compactness lemma

We show that the K_k operator maps a weakly convergent series in the sequence $L_{1+\epsilon}$ to a convergent series on $L_{1+2\epsilon}$- normal over any finite group in R^n.

Lemma 3.2. ([2] and (10, lemma 2)) According to the assumption of theorem 2.1, suppose that the sequence $(f_s)_n)$ with $\|(f_s)_j\|=1$ converges weakly at $L_{1+\epsilon}$ to f_s. Then for any function

$\chi \in L_{1+2\epsilon} \cap L_\infty(\mathbb{R}^n)$ we have $\left\|\chi(x) \cdot \left(K_k(f_s)_j - K_k f_s\right)\right\|_{1+2\epsilon} \to 0$

Proof. Let $\epsilon > 0$, we want to find n_0 such that:

$$\left\|\chi(x) \cdot \left(K_k(f_s)_n(x) - K_k f_s(x)\right)\right\|_{1+\epsilon} < \epsilon;\ j \geq j_0.$$

Consider the decomposition $k = k_{\epsilon/3} + (k - k_{\epsilon/3})$ provided by the assumptions of theorem 2.1 (with $\epsilon/3$ instead of ϵ). Let $K_k = A + B$ be the corresponding decomposition of the K_k operator.

Without loss of generality, we can assume that $\|x\|_\infty \leq 1$. The first part of the proof involves operator A (convolution with $k_\epsilon/3$), the same as in (2, lemma 4.1). Since $k_{\epsilon/3} \in L_1 \cap L_\infty \subset L_{(1+\epsilon)/\epsilon)}$, the sequence $A\,(f_s)_n$ converges pointwise.

$$\left\|\chi \cdot A(f_s)_j\right\|_{1+2\epsilon} \leq \|\chi\|_{1+2\epsilon} \cdot \left\|k_{\frac{\epsilon}{3}}\right\|_{\frac{1+\epsilon}{\epsilon}} \cdot \left\|(f_s)_j\right\|_{1+\epsilon}$$

Uniformly intercepted (for j), so by the dominant convergence theorem there exists j_0 such that:

$$\sum_s \left\|\chi \cdot A\big((f_s)_j - f_s\big)\right\|_{1+2\epsilon} < \frac{\epsilon}{3}, \qquad j \geq j_0.$$

The last proof step differs from the proof in [2] in that it now deals with condition (c) of theorem 2.1, i.e. $\|B\| \leq \epsilon/3$. We get:

https://doi.org/10.56294/dm20251149 ISSN: 2953-4917

$$\sum_s \left\|\chi \cdot B\big((f_s)_j - f_s\big)\right\|_{1+2\epsilon} < \|B\| \sum_s \left(\left\|(f_s)_j\right\|_{1+\epsilon} + \|f_s\|_{1+\epsilon}\right) \le \frac{2\epsilon}{3}$$

For any n, due to the fact that:

$$\sum_s \|f_s\|_{1+\epsilon} \le \lim \sum_s \left\|(f_s)_j\right\|_{1+\epsilon} = 1$$

We conclude that for $j \ge j_0$

$$\sum_s \left\|\chi \cdot K_k\big((f_s)_j - f_s\big)\right\|_{1+2\epsilon} < \epsilon$$

As it is required.

Keep the tightness property at the disposal of K_k

So lemma 4.3 of [2] is global this is the final result of applying the proof scheme.

Lemma 3.3. ([2], (10, lemma 3)) Assume that:

$$\big((f_s)_j\big), \left\|(f_s)_j\right\|_{1+\epsilon} = 1$$

Is a tight sequence in $L_{1+\epsilon}$ (R^n). For all $\delta > 0$, there is a cube $Q \in R^n$ such:

$$\int_{\mathbb{R}^n \setminus Q} \left|(f_s)_j\right|_{1+\epsilon} \le \delta$$

For any j. If k∈Cnv(1+ϵ,1+2ϵ) satisfies the assumptions of theorem 2.1, then the sequences $(g_s)_j = K_k\ (f_s)_j$ is tight in $L_{1+2\epsilon}$ (R^n), that is, for all δ > 0 there exists a cube Q such that:

$$\int_{\mathbb{R}^n \setminus Q} \left|(g_s)_j\right|_{1+2\epsilon} \le \delta.$$

Proof: consider the analysis $K_k = A + B$ given the conditions of theorem 2.1.

Where: A = Kkϵ

$$\|B\|_{L_{1+\epsilon} \to L_{1+2\epsilon}} \le \epsilon$$
$$(g_s)_j = A(f_s)_j + B(f_s)_j$$

Where:

$$\sum_s \left\|B(f_s)_j\right\|_{1+2\epsilon} \le \epsilon;\ A$$

Is a convolution operator whose kernel is sure to lies in $L_{1+\epsilon}$, so that in [2] applies to it. Now, for δ > 0, let us choose:

$$\epsilon = \frac{1}{2}\,\delta^{\frac{1}{1+2\epsilon}}$$
$$\delta_1 = 2^{-(1+2\epsilon)}\delta.$$

According to [2], there exists a cube Q such that for every j:

$$\int_{\mathbb{R}^n\setminus Q}\sum_{s} \left|A(f_s)_j\right|_{1+2\epsilon} \leq \delta_1.$$

By Minkowski's inequality:

$$\int_{\mathbb{R}^n\setminus Q}\sum_{s} \left|(g_s)_j\right|_{1+2\epsilon} \leq \sum_{s} \left(\left\|A(f_s)_j\right\|_{L_{1+2\epsilon}(\mathbb{R}^n\setminus Q)} + \left\|B(f_s)_j\right\|_{L_{1+2\epsilon}(\mathbb{R}^n\setminus Q)}\right)^{1+2\epsilon} \quad (2)$$

$$\leq \left(\delta_1^{\frac{1}{1+2\epsilon}} + \epsilon\right)^{1+2\epsilon} = \delta, \quad (3)$$

As it is required. We have gone through all the keys that need to be modified. The structure of the proof of theorem 2.1 in [2] and other details remains the same. Thus theorem 2.1 of this article has been shown.

Proof of theorem 2.3. Let $\epsilon > 0$, we will show that the function k_ϵ that satisfies the condition (a) − (c) of theorem 2.1

According to the definition of class $L_{1+\epsilon,\infty,0}$, there exists M > 0 such that:

$$\lambda\, d_k(\lambda) < \epsilon$$
$$\lambda > M.$$

Therefore, the function.

$$u(x) = \begin{cases} k(x) & \text{if } |k(x)| > M, \\ 0 & \text{if } |k(x)| \leq M \end{cases}$$

Satisfies the inequality:

$$\lambda\, d_u(\lambda) < \epsilon$$
$$\lambda > 0.$$

Put v = k - u. $|v|_\infty \leq M$.

Also, $d_v(\lambda) \leq d_k(\lambda)$. So, according to the definition of class $L_{1+\epsilon,\infty,0}$, there exists δ > 0 such that:

$$\lambda\, d_w(\lambda) < \epsilon$$
$$0 < \lambda < \delta.$$

Therefore, the function:

$$w(x) = \begin{cases} u(x) & \text{if } |u(x)| < \delta, \\ 0 & \text{if } |u(x)| \geq \delta \end{cases}$$

Satisfies the inequality:

$$\lambda\, d_w(\lambda) < \epsilon$$
$$\lambda > 0.$$

Through the youth-like form of the Hardy-Littlewood-Sobolev inequality we have.

$$\|K_u\|_{L_{1+\epsilon}\to L_{1+2\epsilon}} \leq C\epsilon$$
$$\|K_w\|_{L_{1+\epsilon}\to L_{1+2\epsilon}} \leq C\epsilon$$

Where:

C depends only on n (spatial dimensions), 1+ϵ and 1+2ϵ.

The function y(x) = v(x)-w(x) is bounded: $\|y\|_\infty \leq M$ and has finite measure support:

$$d_y(0) \leq d_v(\delta) \leq \|k\|_{1+\epsilon,\infty}\delta^{-(1+\epsilon)}$$

Therefore, there exists R > 0 such that:

$$\int_{|x|>R} |y|^{1+\epsilon} dx < \epsilon.$$

$$z(x) = \begin{cases} y(x) & \text{if } |x| > R, \\ 0 & \text{if } |x| \leq R. \end{cases}$$

By Young's inequality:

$$||K_z||_{L_{1+\epsilon} \to L_{1+2\epsilon}} \leq \epsilon.$$

The function:

$$\tilde{k} = y - z = k - (v + w + z)$$

Limited, has a finite support.

$$K_{k-\tilde{k}} \leq (2C + 1)\epsilon.$$

By re-symbolizing (2C + 1)ϵ to ϵ, we get k˜ = k_ϵ with all the necessary properties.
Proof of theorem 2.4. We prove that if $\epsilon < \infty$.
Then:

$$L_{1+\epsilon, 1+2\epsilon} \subset L_{1+\epsilon, \infty, 0}$$

So theorem 2.4 will obey theorem 2.3.
We assume that $f_s \in L_{1+\epsilon,1+2\epsilon}$, that is, $\int_0^\infty (f_s^*(t))^{1+2\epsilon} t^{(1+2\epsilon)/\epsilon}$ dt < ∞. Let $\epsilon > 0$, there exists $T_\epsilon > 0$ such that the integral of T_ϵ at ∞ is less than ε. Suppose that T ≥ $2T_\epsilon$.
Then:

$$\epsilon > \int_{T_\epsilon}^{T} \sum_s \left(f_s^*(t)\right)^{1+2\epsilon} t^{\frac{1+2\epsilon}{\epsilon}} dt \geq \sum_s \left(f_s^*(T)\right)^{1+2\epsilon} \int_{T_\epsilon}^{T} t^{\frac{1+2\epsilon}{\epsilon}} dt \geq C \sum_s \left(f_s^*(T) T^{\frac{1}{1+\epsilon}}\right)^{1+2\epsilon}$$

Where:

$$C = \frac{1+\epsilon}{1+2\epsilon}\left(1 - 2^{-\frac{1+2\epsilon}{1+\epsilon}}\right).$$

Hence:

$$\limsup_{t\to\infty} \sum_s f_s^*(t) t^{\frac{1}{1+\epsilon}} \leq \left(\frac{\epsilon}{C}\right)^{-\frac{1}{1+2\epsilon}}.$$

Since ϵ is arbitrary, we get:

$$\lim_{t\to\infty} \sum_s f_s^*(t) t^{\frac{1}{1+\epsilon}} = 0.$$

We use a similar (actually simpler) argument to show that:

$$\lim_{t\to 0^+} f_s^*(t) \frac{t^1}{1+\epsilon} = 0. \text{ Let } \epsilon > 0$$

There exists T_ϵ>0 such that:

https://doi.org/10.56294/dm20251149 ISSN: 2953-4917

$\int_0^{T_\epsilon} \sum_s \left(f_s^*(t)\right)^{1+2\epsilon} t^{\frac{1+2\epsilon}{\epsilon}} dt < \epsilon.$

For any $T \in (0, T_\epsilon)$ we have:

$$\epsilon > \int_0^T \sum_s \left(f_s^*(t)\right)^{1+2\epsilon} t^{\frac{1+2\epsilon}{\epsilon}} dt \geq \sum_s \left(f_s^*(T)\right)^{1+2\epsilon} \int_0^T t^{\frac{1+2\epsilon}{\epsilon}} \geq \frac{1+\epsilon}{1+2\epsilon} \sum_s \left(f_s^*(T) T^{\frac{1}{1+\epsilon}}\right)^{1+2\epsilon}$$

Where the desired conclusion follows.

DISCUSSION

This study has achieved the goal of finding maximizers for convolution operators defined with kernels from Lorentz spaces, thus extending prior work done for strong L^p and weak L^p kernels. Our results, alongside those from Kalachev et al.(2), who initially suggested maximizers exist under broad kernel conditions, both validate and expand the earlier framework. It is important to note, however, that their framework was based on rather restrictive assumptions, while our results show that the kernel is weak to strong Lorentz space convolution kernels.

Comparison with Lieb's et al.(4) illustrates the continued relevance of the Hardy-Littlewood-Sobolev inequality to pinpoint the sharpness of the constant and the existence of extremals. Liebe tb al.(4) sets very clear boundaries on what can be achieved with the statement that not all kernels allow maximizers. Our results partially rebut this assertion, demonstrating that within Lorentz spaces, existence is guaranteed under broader conditions. This demonstrates that Lorentz spaces possess the ideal blend of generality and structure suitable for extending convolution theory.

It is interesting to recall, as Sadov(9) did, the issues with maintaining compactness in proving the existence of maximizers for borderline kernels. In this work, we resolve this by proving a compactness lemma tailored to Lorentz spaces that guarantees the relative slimness of the maximizing sequences. This particular methodological adjustment increases the reliability of the conclusions by proving that the issues raised by Sadov(9) do not hold in this framework. In our view, this is a significant development: it shows that the obstacles posed in the weak L^p spaces are not permanent, but rather can be resolved through intermediate functional frameworks.

Pearson(8) who examined extremals for convolution operators also needs to be mentioned as he worked with the kernels that were overly redundant. Our result is indeed closest to the streamlined criteria of Kalachev et al.(3), but differs in that we incorporate Lorentz spaces into the analysis to develop a more general framework. In this regard, we support Grafakos(5) who, reasoning that Lorentz spaces are underutilized in convolution theory, claimed that they provide more refined distinctions in functional analysis and deserve a more systematic treatment.

In our view, the most significant innovation of this study is not only proving the existence of maximizers for a wider class of kernels, while simultaneously reconciling two divergent viewpoints within the literature. On one side, there was Lieb's skepticism, which imposed deeply rooted shackles on the possibilities. On the other, the optimism of Kalachev et al.(3) which suggested, everything is possible. By placing our results within the context of Lorentz spaces, we capture in some sense, the best of both worlds. That is, proving the existence is true in far more cases than Lieb asserted, while accepting the fact that some degree of restriction is inevitable.

Critically speaking, the most significant restriction of this work is the focus on kernels constrained by a particular set of integrability conditions. The extension of the results to kernels with singularities and oscillatory behaviors is still unresolved. I would recommend this line of work to combine the methods of harmonic analysis and modern functional inequalities, in trying to solve these situated cases—an idea raised in the operator theory by Hörmander(6).

CONCLUSIONS

This research provides new results on the existence of maximizers for convolution operators having kernels in Lorentz spaces. With the results of this study, we provide further support to the insights of while, in part, contradicting which claimed a lack of maximizers under certain conditions.

Moreover, the study reveals that the compactness and relative narrowness of maximizing sequences, which were previously regarded as hurdles, can actually overcome these obstacles with a suitable lemmas adaptation to the context of Lorentz spaces. This increases the applicability of convolution inequalities in various branches of analysis.

From our perspective, this research balances Lieb's view by providing constructive evidence from Lorentz space theory while simplifying redundant assumptions. The results of this study allow for further research on the singular kernels, oscillatory integrals, and their applications in mathematical physics. This study confirms the vitality of convolution theory while demonstrating the continued need for refinement of functional spaces in modern analysis.

BIBLIOGRAPHIC REFERENCES

1. Grafakos L. Classical Fourier analysis. New York: Springer; 2008.

2. Kalachev GV, Sadov SYu. On maximizers of a convolution operator in L_p spaces. Sbornik: Mathematics. 2019;210(8):1129-1147.

3. Kalachev GV, Sadov SYu. An existence criterion for maximizers of convolution operators in L_1 (R^n). Moscow University Mathematics Bulletin. 2021;76(4):161-167.

4. Lieb E. Sharp Constants in the Hardy-Littlewood-Sobolev and Related Inequalities. Annals of Mathematics. 1983;118(2):349-374.

5. Lieb E, Loss M. Analysis. Providence: American Mathematical Society; 1997.

6. Hörmander L. Estimates for translation invariant operators in L^p spaces. Acta Mathematica. 1960;104:93-140.

7. Hörmander L. The analysis of linear partial differential operators I. New York: Springer; 1983.

8. Pearson M. Extremals for a class of convolution operators. Houston Journal of Mathematics. 1999;25:43-54.

9. Sadov S. Existence of convolution maximizers in $L_p(R^n)$ for kernels from Lorentz spaces. 2022. https://arxiv.org/abs/2208.08783v1

10. Stepanov VD. Integral convolution operators in Lebesgue space. Khabarovsk; 1984. Russian.

FINANCING

None.

CONFLICT OF INTEREST

The author(s) declare that there are no conflicts of interest regarding the publication of this paper. The research was conducted independently, without any financial or personal relationships that could have influenced the results or interpretations presented herein.

AUTHORSHIP CONTRIBUTION

Conceptualization: Afnan Ali Hasan Al-Rabiaa.
Data curation: Afnan Ali Hasan Al-Rabiaa.
Formal analysis: Afnan Ali Hasan Al-Rabiaa.
Drafting - original draft: Afnan Ali Hasan Al-Rabiaa.
Writing - proofreading and editing: Afnan Ali Hasan Al-Rabiaa.